Representation varieties & TQFTs

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Representation varieties

X = connected closed manifold

$\pi_1(X)$ = fundamental group

G = algebraic group over k

$\mathfrak{X}_G(X) = \text{Hom}(\pi_1(X), G)$ <u>G-representation variety of X</u>

When C complex projective curve

$$
\begin{array}{ccc}
 & \mathcal{M}_{\mathsf{B}}(C, G) & \\
\text{Riemann–Hilbert} \quad \swarrow\!\!\!\nearrow & & \nwarrow\!\!\!\searrow \quad \text{Hitchin–Kobayashi} \\
\mathcal{M}_{\mathsf{DR}}(C, G) & \longleftrightarrow & \mathcal{M}_{\mathsf{Dol}}(C, G)
\end{array}
$$

E-polynomial $e(X) = \sum_{k,p,q} (-1)^k \; h_c^{k;p,q}(X) \; u^p v^q \in \mathbb{Z}[u, v]$

Goal: find class of $\mathfrak{X}_G(\Sigma_g)$ in $\mathsf{K}(\mathbf{Var}_k)$

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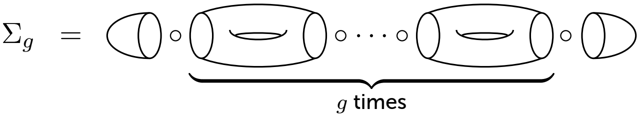
- Counting points over finite fields
- Geometric method $\Rightarrow$ Topological Quantum Field Theories (TQFTs)

Idea: cut manifold in pieces and 'compute invariant piecewise'



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Bordism category $\mathbf{Bdp}_n$

- Objects (M, A)
- Morphisms (W, A)

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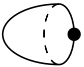
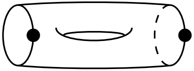
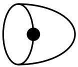
Examples in dimension $n = 2$:



$D^\dagger : (S^1, \star) \to \varnothing$



$L : (S^1, \star) \to (S^1, \star)$



$D : \varnothing \to (S^1, \star)$

$$\textbf{Bdp}_n \xrightarrow{\mathcal{F}} \mathsf{Span}(\textbf{Var}_k) \xrightarrow{\mathcal{Q}} \mathsf{K}(\textbf{Var}_k)\text{-}\mathsf{Mod}$$

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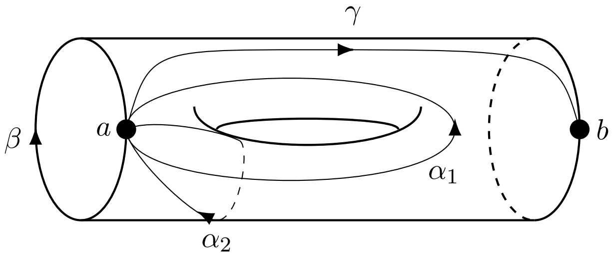
- Define $\mathcal{F}(M, A) = \mathfrak{X}_G(M, A)$
- Given $(W, A) : (M_1, A_1) \to (M_2, A_2)$, define

$$\mathcal{F}(W, A) : \;\; \mathfrak{X}_G(M_1, A_1) \longleftarrow \mathfrak{X}_G(W, A) \longrightarrow \mathfrak{X}_G(M_2, A_2)$$

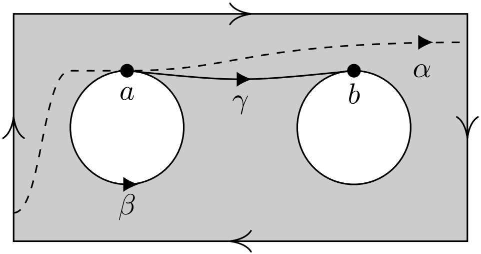
- Define $\mathcal{Q}(X) = \mathsf{K}(\mathbf{Var}/X)$
- Define $\mathcal{Q}(X \xleftarrow{f} Z \xrightarrow{g} Y) = g_! \circ f^*$

Useful: Z produces invariants



$$L : (S^1, \star) \to (S^1, \star)$$

$$\mathcal{F}(L): \quad \begin{array}{ccccc} G & \longleftarrow & G^4 & \longrightarrow & G \\ B & \longleftarrow & (B, A_1, A_2, C) & \longmapsto & CB[A_1, A_2]C^{-1} \end{array}$$



$$N : (S^1, \star) \to (S^1, \star)$$

$$\mathcal{F}(N) : \qquad \begin{array}{ccccc} G & \longleftarrow & G^3 & \longrightarrow & G \\ B & \longleftarrow\!\shortmid & (B, A, C) & \longmapsto & CBA^2C^{-1} \end{array}$$

So $Z(L), Z(N)$ are maps $\mathsf{K}(\textbf{Var}/G) \to \mathsf{K}(\textbf{Var}/G)$

$$[\mathfrak{X}_G(\Sigma_g)] \;=\; \frac{1}{[G]^g}\;\; Z(D^\dagger) \;\circ\; Z(L)^g \;\circ\; Z(D)(1)$$

Computed

$Z(L)$ for $G = \mathbb{U}_2, \mathbb{U}_3, \mathbb{U}_4$ and $Z(N)$ for $G = \mathbb{U}_2, \mathbb{U}_3$

For $G = \mathbb{U}_2$, we have $Z(L) = \begin{bmatrix} q^2(q-1) & q^2(q-2) \\ q^2(q-2)(q-1) & q^2(q^2-3q+3) \end{bmatrix}$

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$$\begin{bmatrix} (q-1)^2(q^2+q-1) & q^2(q-2)^2 & q^2(q-2)(q-1) & q^2(q-2)(q-1) & (q-1)^3(q+1) \\ q^3(q-2)^2(q-1)^2 & q^3(q^2-3q+3)^2 & q^3(q-2)(q-1)(q^2-3q+3) & q^3(q-2)(q-1)(q^2-3q+3) & q^3(q-2)^2(q-1)^2 \\ q^3(q-2)(q-1)^2 & q^3(q-2)(q^2-3q+3) & q^3(q-1)(q^2-3q+3) & q^3(q-2)^2(q-1) & q^3(q-2)(q-1)^2 \\ q^3(q-2)(q-1)^2 & q^3(q-2)(q^2-3q+3) & q^3(q-2)^2(q-1) & q^3(q-1)(q^2-3q+3) & q^3(q-2)(q-1)^2 \\ (q-1)^4(q+1) & q^2(q-2)^2(q-1) & q^2(q-2)(q-1)^2 & q^2(q-2)(q-1)^2 & (q-1)^2(q^3-q^2+1) \end{bmatrix}$$

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For $G = \mathbb{U}_4$, we have $Z(L) =$

[illegible]

Results

$$[\mathfrak{X}_{\mathbb{U}_2}(\Sigma_g)] = q^{2g-1}(q-1)^{2g+1}((q-1)^{2g-1}+1)$$

$$\begin{aligned}[\mathfrak{X}_{\mathbb{U}_3}(\Sigma_g)] = {} & q^{3g-3}(q-1)^{2g}\left(q^2(q-1)^{2g+1}\right. \\ & \left. + q^{3g}(q-1)^2 + q^{3g}(q-1)^{4g} + 2q^{3g}(q-1)^{2g+1}\right)\end{aligned}$$

$$[\mathfrak{X}_{\mathbb{U}_4}(\Sigma_g)] = \ldots$$

$$[\mathfrak{X}_G(N_r)] = \ldots$$

Stacky TQFT

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Replace $$\mathsf{K}(\mathbf{Var}_k) \longmapsto \mathsf{K}(\mathbf{Stck}_{\mathsf{BG}})$$
$$\mathfrak{X}_G(X) \longmapsto [\mathfrak{X}_G(X)/G]$$

to obtain $$Z : \mathbf{Bdp}_n \to \mathsf{K}(\mathbf{Stck}_{\mathsf{BG}})\text{-}\mathbf{Mod}$$

For $G = \mathsf{AGL}_1(\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \right\}$

$$Z(L) = \begin{bmatrix} [\mathbb{G}_a/G]q\,(q-2) + [\mathbb{G}_m/G]\,(q+1) + 1 & [\mathbb{G}_a/G][\mathbb{G}_m/G]q\,(q-2) \\ [\mathbb{G}_a/G]q\,(q-2) & [\mathbb{G}_a/G][\mathbb{G}_m/G]q\,(q-2) + q^2 \end{bmatrix}$$

E.g. for $g = 3$,

$$
\begin{aligned}
[\mathfrak{X}_G(\Sigma_3)/G] = 1 \\
&+ [\mathbb{G}_a/G] q \left(q - 2\right) \left(q^2 - 3q + 3\right) \left(q^2 - q + 1\right) \\
&+ [\mathbb{G}_m/G] \left(q + 1\right) \left(q^2 - q + 1\right) \left(q^2 + q + 1\right) \\
&+ [G/G] q \left(q - 2\right) \left(q + 1\right) \left(q^2 + 1\right) \left(q^2 - 3q + 3\right) \left(q^2 - q + 1\right)
\end{aligned}
$$